

An algorithm for finding Hamiltonian Cycles in Cubic Planar Graphs

Bohao Yao
 Charl Ras, Hamid Mokhtar
 The University of Melbourne

Abstract

We first prove a one-to-one correspondence between finding Hamiltonian cycles in a cubic planar graphs and finding trees with specific properties in dual graphs. Using this information, we construct an exact algorithm for finding Hamiltonian cycles in cubic planar graphs. The worst case time complexity of our algorithm is $O(2^n)$.

1 Introduction

A *Hamiltonian cycle* is a cycle which passes through every vertex in a graph exactly once. A *planar graph* is a graph which can be drawn in the plane such that no edges intersect one another. A *cubic graph* is a graph in which all vertices have degree 3. Finding a Hamiltonian cycle in a cubic planar graph is proven to be an \mathcal{NP} -Complete problem [2]. This implies that unless $\mathcal{P} = \mathcal{NP}$, we could not find an efficient algorithm for this problem. Most approaches to finding a Hamiltonian cycle in planar graph utilises the *divide-and-conquer* method, or its derivation, the *separator theorem* which partitions the graph in polynomial time [4]. Exact algorithms using such methods were found to have the complexity of $O(c^{\sqrt{n}})$ [3] [1], where n denotes the number of vertices and c is a constant. In this paper, we consider only cubic planar graphs and attempt to find a new algorithm to provide researchers with a new method to approaching this problem.

2 The Expansion Algorithm

We first start by introducing our so-called *Expansion Algorithm* which increases the number of vertices in a cycle at each iteration. A cycle can be first found by taking the outer facial cycle of the planar graph. We define it as the *base cycle*, σ_0 . This base cycle is then expanded by the Expansion Algorithm, which will be described in detail later.

Definition 2.1. Consider a planar graph $G = (V, E)$

A complementary path, P_e^σ , is a path between 2 adjacent vertices, $v_1, v_2 \in \sigma$ connected by the edge, e , s.t. P_e^σ is internally disjoint from σ .

Furthermore, P_e^σ and e together will form the boundary of a face in G .

Assuming we are not dealing with multigraphs, the complementary path will always have at least one other vertex besides v_1, v_2 .

The restriction that P_e^σ and e have to form the boundary of a face will be used later to prove Corollary 2.6.

Definition 2.2. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Then, $G_1 + G_2 := (V_1 \cup V_2, E_1 \cup E_2)$

Algorithm 2.3 (Expansion Algorithm).

```

Let  $\sigma_0$  be the outer facial cycle.
Let  $i = 0$ ;
if  $\exists P_{e_i}^{\sigma_i}$  then
    |  $\sigma_{i+1} := \sigma_i + P_{e_i}^{\sigma_i} - e_i$ ;
    |  $i = i + 1$ ;
else
    | Output  $\sigma_i$ 
end

```

At each iteration, the algorithm removes an edge, e and adds a path P_e^σ . Since there are no vertices on e and there is at least 1 vertex on P_e^σ , the number of vertices on the cycle will always increase at each iteration. Since there is only a finite amount of vertices in the graph, the algorithm will have to terminate eventually.

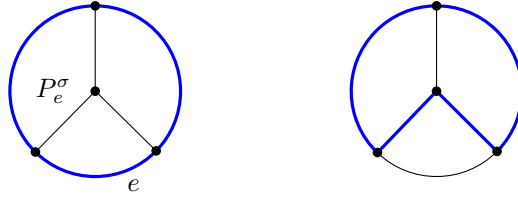


Figure 1: Example of utilizing the Expansion Algorithm with the base cycle in blue

Definition 2.4. The interior of a cycle, C , is the connected region lying to the left of an anticlockwise orientation of C .

Lemma 2.5. At each iteration of the Algorithm 2.3, all the vertices of G either lie in the interior of σ , or on σ .

Proof. Let P_i be the statement that all the vertices of G either lie in the interior of σ_i

- P_0 is true as σ_0 is the outer facial cycle, with all the vertices either lie in the interior or on σ_0 .
- Assume P_k true. Then all the vertices either lie in the interior or on σ_k . Assume $\exists P_{e_k}^{\sigma_k}$, then σ_{k+1} exists. Let f_k be the face bounded by $P_{e_k}^{\sigma_k}$ and e_k . The interior of f_k originally lies inside σ_k . But by an iteration of Algorithm 2.3, the interior of f_k now lies outside the new base cycle, σ_{k+1} . However, since there are no vertices in the interior of a face, all the vertices in σ_{k+1} also lies either in the interior or on σ_{k+1} . Therefore, if σ_{k+1} exists, then $P_k \Rightarrow P_{k+1}$.

Hence, by mathematical induction, P_i is true $\forall i$ as long as σ_i exists. \square

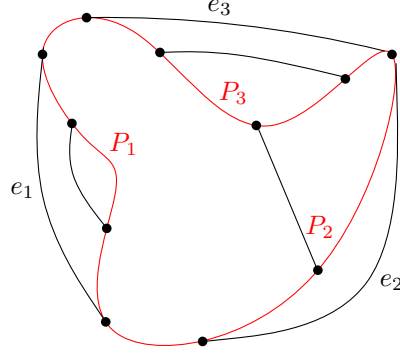


Figure 2: Illustration of the proof for Lemma 2.7 with C in red

Corollary 2.6. *If P_e^σ exists for an edge $e \in \sigma$, then P_e^σ is unique.*

Proof. By Lemma 2.5, none of the vertices can lie outside σ , thus, P_e^σ also lies in the interior of σ . Any edge e lies on the boundary of 2 faces. If $e \in \sigma$, then \exists only 1 possible P_e^σ such that e and P_e^σ forms the boundary of a face that lies in the interior of σ (the other face that e is a boundary of lies outside σ) \square

Lemma 2.7. *If G have a Hamiltonian cycle, then \exists a choice of complementary paths that algorithm 2.3 can use to find that Hamiltonian cycle.*

Proof. Consider a Hamiltonian cycle, C , in G . If $C = \sigma_0$ then the case is trivial. If $C \neq \sigma_0$, then by Lemma 2.5, all vertices in C lies either on or in the interior of σ_0 since C contains all the vertices of the graph. Since C contains all the vertices and σ_0 is a cycle and therefore must contain at least 3 vertices, $C \cap \sigma_0 \neq \emptyset$.

Suppose $C \neq \sigma_0$. Let v_1, v_2, \dots, v_n be consecutive vertices on σ_0 about the clockwise rotation. Let P_i be the path connecting v_i, v_{i+1} that is the subpath of C . Let the edge between v_i and v_{i+1} be e_i which also lies on σ_0 . Since C contains all the vertices in the graph, by iteratively finding P_e^σ , starting with $e = e_i$, more subpath of P_i will lie on σ at each iteration. The algorithm will only move on when $P_i \subset \sigma$.

Repeating this process $\forall i$, we will eventually end up with $\sigma = C$, in which the algorithm will terminate. \square

3 The Problem in the Dual Graph

Definition 3.1. *Given a cubic planar graph, G , and the corresponding dual graph, $\overline{G} = (\overline{V}, \overline{E})$. A corresponding face, $f_{\overline{v}} \in G$, is the face corresponding to the vertex $\overline{v} \in \overline{G}$.*

Definition 3.2. *The outer vertex, \overline{v}^* , is the vertex in \overline{G} that corresponds to the outer face of G .*

Definition 3.3. *Let $e \in G$ be the shared boundary between $f_{\overline{v}_1}, f_{\overline{v}_2}$. A dual edge, $\overline{e} \in \overline{G}$, is defined as an edge between $\overline{v}_1, \overline{v}_2 \in \overline{G}$.*

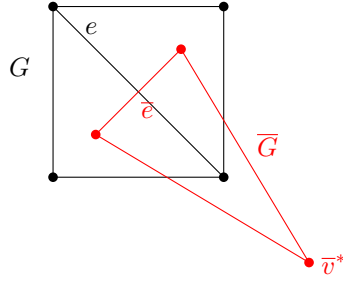


Figure 3: Example of \bar{e} and \bar{v}^*

Consider Algorithm 2.3. Since P_e^σ is unique for an edge e by Corollary 2.6, let e_0, e_1, \dots, e_n be the edges chosen in the Expansion Algorithm in order. Let $\bar{e}_0, \bar{e}_1, \dots, \bar{e}_n$ be the corresponding dual edges in \bar{G} .

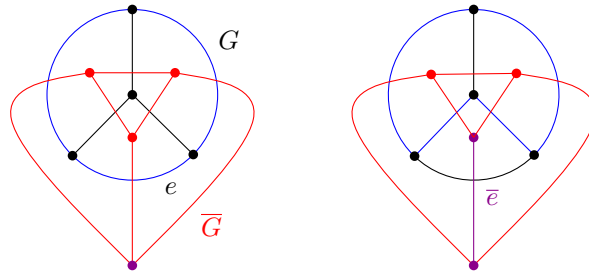


Figure 4: Expansion Algorithm in the Dual Graph with \bar{T} in Purple

Lemma 3.4. $\bigcup_{i=0}^n \bar{e}_i$ is a tree (\bar{T}).

Proof. Let P_a be the statement that $\bigcup_{i=0}^a \bar{e}_i$ is a tree.

- P_0 is true as 2 vertices connected by an edge, \bar{e}_0 is considered a tree.
- Assume P_k is true. Then $\bigcup_{i=0}^k \bar{e}_i$ is a tree. From proof of Corollary 2.6, e_{k+1} is a boundary of 2 faces, one that lies inside σ_{k+1} (f_1) and another that lies outside σ_{k+1} (f_2). From proof of Lemma 2.5, by including path $P_{e_i}^\sigma$, the face bounded by e_i and $P_{e_i}^\sigma$ which originally lies in the interior of σ_i will now lie on the exterior of σ_{i+1} . Therefore, the vertex in \bar{G} that corresponds to f_1 will lie outside \bar{T} while the vertex that corresponds to f_2 will lie inside \bar{T} . Thus, \bar{e}_{k+1} will connect a vertex on \bar{T} to a vertex that is outside of \bar{T} , thus expanding the tree. Hence $\bigcup_{i=0}^{k+1} \bar{e}_i$ is also a tree and therefore P_{k+1} is true. Thus, $P_k \Rightarrow P_{k+1}$.

By mathematical induction, $\bigcup_{i=0}^n \bar{e}_i$ is a tree. □

From Lemma 2.5, we know that faces f_k , bounded by e_k and $P_{e_k}^{\sigma_k}$, will be on the exterior of σ if e_k was used in the algorithm. Conversely, if e_k was not used in the algorithm, then f_k will be in the interior of the cycle σ . This is equivalent to stating that all the vertices in \overline{T} lies outside σ while all the vertices not in \overline{T} lies in the interior or σ .

Lemma 3.5. *If $\overline{v} \in \overline{G}$ lies on \overline{T} , then all the vertices in G on $f_{\overline{v}}$ lies on σ_n .*

Proof. If \overline{v} lies on the tree, then $\exists P_e^\sigma$ s.t. all its vertices in G on $f_{\overline{v}}$ lies on P_e^σ since the face is bounded by P_e^σ and e . Since each iteration of the algorithm will only remove an edge and not any vertices, all the vertices already on the cycle will remain on the cycle at each iteration. \square

Theorem 3.6. *G has a Hamiltonian cycle if and only if $\exists \overline{T}$ found by Algorithm 2.3, with \overline{v}^* at the root, that satisfy the following properties:*

1. *For each vertex $v \in G$ that lies on the boundaries of $f_{\overline{v}_1}, f_{\overline{v}_2}, f_{\overline{v}_3}$, at least one of the vertices, $\overline{v}_1, \overline{v}_2, \overline{v}_3 \in \overline{G}$, lies on \overline{T} .*
2. *No two vertices on \overline{T} can be joined by an edge \overline{e} unless $\overline{e} \in \overline{T}$.*

Proof. By first taking the outer facial cycle, σ_0 , only \overline{v}^* lies outside our base cycle. Hence, \overline{T} starts with \overline{v}^* as its root.

(\Rightarrow) We will begin by proving each of the following property:

1. If G has a Hamiltonian cycle, then \exists a cycle in G that uses every vertex in G . If none of the vertices, $\overline{v}_1, \overline{v}_2, \overline{v}_3$, lies on \overline{T} , then none of the P_e^σ found by Algorithm 2.3 could contain v from negation of Lemma 3.5. This contradicts the definition of a Hamiltonian cycle. Therefore, at least one of the vertices lies on \overline{T} .

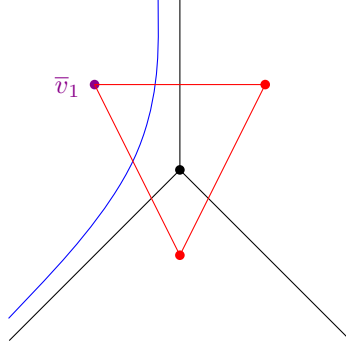
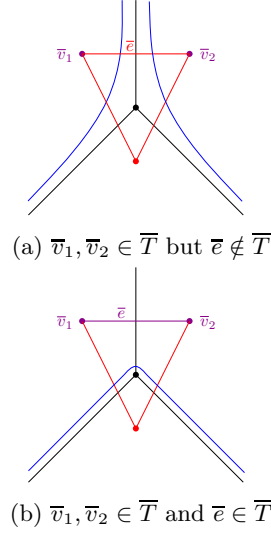


Figure 5: A vertex lies on the cycle (blue) as $\overline{v}_1 \in \overline{T}$ (purple)

2. If 2 vertices, $\overline{v}'_1, \overline{v}'_2 \in \overline{T}$ are adjacent, \exists 2 vertices shared by $f_{\overline{v}'_1}, f_{\overline{v}'_2} \in G$. However, the edge between them is not used on \overline{T} . This implies that the 2 vertices will lie on the corresponding path in G twice, once from each face $f_{\overline{v}'_1}, f_{\overline{v}'_2} \in G$. This is a contradiction from the definition of a Hamiltonian cycle.



(\Leftarrow) Suppose that $\bar{T} = \bigcup_{i=0}^n \bar{e}_i$ is a tree found by Algorithm 2.3 that \bar{T} satisfies those properties.

From 2, we apply restriction on \bar{T} such that the path in G passes through each vertex at most once (as proven previously). From 1, we know the path passes through every vertex in G at least once.

Since there is a path in G that passes through every vertex at most once and at least once, we know the path passes through every vertex in G once. Furthermore, we know that Algorithm 2.3 will always output a cycle, σ . Therefore, we know that \bar{T} satisfying those properties will correspond to a Hamiltonian cycle in \bar{G} \square

Note that Theorem 3.6 is a corollary of a theorem by Skupień [5].

4 Finding \bar{T}

We start solving the problem by modifying a backtracking algorithm (**procedure solve**) with restrictions (**procedure update**) to limit the amount of guessing required. Restrictions from **procedure update** may cause the tree to form 2 disjoint graphs, in which case a new backtracking algorithm (**procedure disjoint**) is called to trace a path between the graphs. If a particular choice of paths does not work, the algorithm will negate that choice (**procedure backtrack**). The algorithm will terminate if it successfully find \bar{T} (all vertices assigned) or if all other choices are exhausted, concluding that the graph has no Hamiltonian cycle.

4.1 Search Algorithm

Algorithm 4.1. *To find \bar{T} with properties in Theorem 3.6, we use a modified backtracking algorithm.*

```

procedure update( $\overline{G}, \overline{v}$ ):
if  $\exists$  an unassigned vertex,  $\overline{v}_1$ , adjacent to both  $\overline{v}$  and any other vertex,  $\overline{u} \in \overline{T}$ , s.t.  $\exists$  a path
between  $\overline{u}$  and  $\overline{v}$  ; // See comments (*)
then
|  $\overline{S} := \overline{S} + \overline{v}_1$ 
end
/* Satisfies Property 2 of Theorem 3.6 */
if  $\exists$  2 adjacent vertices,  $\overline{v}_1, \overline{v}_2 \in \overline{S}$ ;
then
|  $V(\overline{T}) := V(\overline{T}) + \overline{v}_3$ , where  $\overline{v}_3$  is all the vertices adjacent to both  $\overline{v}_1$  and  $\overline{v}_2$ ;
end
/* Satisfies Property 1 of Theorem 3.6 */
if  $\exists$  a vertex  $\in \overline{S} \cap \overline{T}$  then
| return false ; // A vertex cannot lie in both  $\overline{S}$  and  $\overline{T}$ 
else
| return true
end

procedure backtrack( $\overline{G}, \overline{v}$ ):
 $V(\overline{T}) := V(\overline{T}) - \overline{v}$ ;
 $\overline{S} := \overline{S} - \overline{v}$ ;
/* If  $\exists$  a contradiction, negate the assumption made about  $\overline{v}$  */

procedure solve( $\overline{G}$ ):
if all vertices  $\in \overline{G}$  assigned then
| return true
else
| for each unassigned vertex,  $\overline{v}$ , adjacent to any vertex  $\overline{w} \in \overline{T}$  do // guess  $\overline{v} \in \overline{T}$ 
| |  $\overline{T} := \overline{T} + (\overline{v}, \overline{w})$  ;
| | if update( $\overline{G}, \overline{v}$ ) = false then // Choice leads to contradiction
| | | return false ;
| | else
| | | if  $\exists$  a disconnected vertex,  $\overline{u} \in \overline{T}$  then
| | | | if disjoint( $\overline{G}, \overline{u}$ ) return false then
| | | | | call backtrack( $\overline{G}, \overline{v}$ )
| | | | end
| | | else
| | | | if solve( $\overline{G}$ ) return true then // Succeeded in finding  $\overline{T}$ 
| | | | | output  $\overline{T}$  ;
| | | | else
| | | | | call backtrack( $\overline{G}, \overline{v}$ )
| | | | end
| | | end
| | end
| end
end

```

```

procedure disjoint( $\overline{G}, \overline{v}$ ):
if all vertices  $\in \overline{G}$  assigned then
  | return true
else
  | if  $\nexists$  a path between  $\overline{v}$  and  $\overline{v}^*$  then
    | for each unassigned vertex,  $\overline{u}$ , adjacent to  $\overline{w} \in \overline{T}$ , s.t.  $\exists$  a path between  $\overline{w}$  and  $\overline{v}$  do
      |  $\overline{T} := \overline{T} + (\overline{u}, \overline{w})$  ; // guess  $\overline{u} \in \overline{T}$ 
      | if update( $\overline{G}, \overline{u}$ ) = false then
        | return false
      | else
        | if disjoint( $\overline{G}, \overline{u}$ ) returns false;
        | then
        | | call backtrack( $\overline{G}, \overline{u}$ )
        | end
      | end
    | end
  | else
    | if solve( $\overline{G}, \overline{v}$ ) returns true;
    | then
    | | output  $\overline{T}$ 
    | else
    | | call backtrack( $\overline{G}, \overline{v}$ )
    | end
  | end
end

```

```

Begin
 $\overline{S} := \emptyset$ ;
 $\overline{T} := \{\overline{v}^*\}$ ;
// Define set  $\overline{S}$  as vertices that cannot lie on  $\overline{T}$ 
call solve( $\overline{G}$ )

```

* Restrictions on vertices does not apply in disjoint graphs where the 2 vertices $\in \overline{T}$ belongs to different sets due to the necessity of connecting the disjoint graphs into one tree.

4.2 Complexity

In the worst case scenario, we expect to force at least one vertex into \overline{S} at each guess. This reduces the number of faces in which we have to check to $\frac{f}{2}$ where f is the number of faces in G . Since we are working in planar graph, $e = \frac{3}{2}n$ where n is the number of vertices. Using Euler's formula:

$$\begin{aligned} n - e + f &= 2 \\ n - \frac{3}{2}n + f &= 2 \\ f &= 2 + \frac{n}{2} \end{aligned}$$

Since we have 2 choices (\overline{T} or \overline{S}) that we have to guess, in the worst case, our algorithm runs in $2^{\frac{f}{2}}$ or $2^{1+\frac{n}{4}}$. Hence, our algorithm has a time complexity of $O(2^n)$.

5 Future Research

We could modify Theorem 3.6 such that it encompasses any planar graph, $G_2 = (V_2, E_2)$. However, caution is advised dealing with faces that share a single vertex, but not a boundary, as this is not reflected in the dual graph, $\overline{G}_2 = (\overline{V}_2, \overline{E}_2)$. To overcome this, we introduce the *imaginary edge*.

Definition 5.1. An imaginary edge, \overline{e}' between vertices in \overline{G}_2 if the corresponding faces in G_2 share a single vertex but not a boundary.

Theorem 5.2. If G_2 has a Hamiltonian cycle then $\exists \overline{T}$ found by Algorithm 2.3, with \overline{v}^* at the root, that at least satisfy the following properties:

1. For each n degree vertex $v \in G_2$ that lies on the boundaries of $f_{\overline{v}_1}, f_{\overline{v}_2}, \dots, f_{\overline{v}_n}$, at least one of the vertices, $\overline{v}_1, \overline{v}_2, \dots, \overline{v}_n \in \overline{G}_2$, lies on \overline{T}
2. No two vertices on \overline{T} can be joined by an edge \overline{e} unless $\overline{e} \in \overline{T}$.
3. No two vertices joined by an imaginary edge can lie on \overline{T}

Proof. Properties 1 and 2 are proven in Theorem 3.6.

The proof of property 3 follows from the proof of Property 2:

If 2 vertices, $\overline{v}_1, \overline{v}_2 \in \overline{T}$ are joined by an imaginary edge, then the vertex in which $f_{\overline{v}_1}$ and $f_{\overline{v}_2}$ shares will be on the cycle twice. Since this contradicts the definition of a Hamiltonian cycle, property 3 holds. \square

6 Acknowledgments

The author wishes to thank his supervisors, Dr Charl Ras and Hamid Mokhtar, for their continued guidance and providing useful insights to this problem.

This paper was supported by the Vacation Research Scholarship awarded by the Australian Mathematical Sciences Institute.

References

- [1] F. Dorn, E. Penninkx, H. L. Bodlaender, and F. V. Foin. Efficient exact algorithms on planar graphs: Exploiting sphere cut branch decompositions. In *Algorithms-ESA 2005*, pages 95-106. Springer, 2005.
- [2] M. R. Garey, D. S. Johnson, and R. E. Tarjan. The planar hamiltonian circuit problem is NP-complete. *SIAM Journal of Computing*, 5(4):704-714, 1976.
- [3] B. Klinz, G. J. Woeginger, et al. Exact algorithms for the Hamiltonian cycle problem in planar graphs. *Operations Research Letters*, 34(3):269-274, 2006.
- [4] R. J. Lipton and R. E. Tarjan. A separator theorem for planar graphs. *SIAM Journal on Applied Mathematics*, 36(2):177-189, 1979.
- [5] Z. Skupień. Hamiltonicity of planar cubic multigraphs. *Discrete mathematics*, 251(1):163-168, 2002.